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UPPER BOUNDS, SECONDARY CONSTRAINTS,
AND BLOCK TRIANGULARITY
IN LINEAR PROGRAMMING

by

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SUMMARY

Short cut computational methods are developed for solving systems whose matrices may be generally described as block triangular.

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With the growing awareness of the potentialities of the linear programming approach to both dynamic and static problems of industry, of the economy, and of the military, the main obstacle toward full application is the inability of current computational methods to cope with the magnitude of the technological matrices for even the simplest situations. However, in certain cases, such as the now classical Hitchcock-Koopmans transportation model, it has been possible to solve the linear inequality system in spite of size because of simple properties of the system [1c]. This suggests that considerable research be undertaken to exploit certain special matrix structures in order to facilitate ready solution of larger systems.

Indeed, recent computational experience has made it clear that standard techniques such as the simplex algorithm, which have been used to solve successfully general systems involving one hundred equations (in any reasonable number of non-negative unknowns), are too tedious and lengthy to be practical for extensions much beyond this figure. Our purpose here will be to develop short cut computational methods for solving an important class of systems whose matrices may be generally described as "block triangular."

Consider a system of equations

$$
\text{(I-1)} \quad \begin{cases} x_o + \sum_{j=1}^{n} a_{oj}x_j = 0 \quad , \quad x_j \geq 0 & (j=1,2,\ldots,n) \;, \\ \sum_{j=1}^{n} a_{ij}x_j = b_i & (i=1,2,\ldots,m) \;, \end{cases}
$$

where it is desired to obtain values of x_j such that the form $\sum a_{oj}x_j$ is to be minimized (or what is the same thing to maximize the variable x_o).

I - Variables with Upper Bounds

The size of matrix associated with such a linear programming problem may become uncomfortably large when, in addition to (I-1), many (or all) variables of the initial set have upper bounds. Thus, if each variable satisfies $0 \leq x_j \leq \alpha_j$, it is customary to add an additional variable, say x_j', and a new equation

$$
\text{(I-2)} \qquad x_j + x_j' = \alpha_j \qquad\qquad (x_j \geq 0, \; x_j' \geq 0)
$$

to take care of each such restriction. To illustrate, by way of example, a linear programming problem of the transportation type involving m destinations and n origins has a matrix involving m+n

rows and $m \cdot n$ variables x_{ij} associated with $m \cdot n$ possible routes joining origins with destinations. Suppose now there is a capacity limitation r_{ij} on a route so that in addition to the original system of equations and linear inequalities $0 \leq x_{ij}$ one must impose $m \cdot n$ additional restraints

$$x_{ij} + x'_{ij} = r_{ij} \qquad (x_{ij} \geq 0, \; x'_{ij} \geq 0).$$

It is clear now the original system has been expanded to $(mn + m + n)$ rows and $2m \cdot n$ variables. Not only has the system become enormously enlarged but it is not clear what has happened to that wonderful triangularity property for transportation type models of a basis which permits ease of hand computation. We shall refer to the original system plus these upper bound restrictions as the enlarged system. The purpose of this section is to show that the upper bound restraints which is a special case of block triangularity (see Section II) may be provided for by applying the simplex algorithm [1b], [3], [8a], to the original system with due care that the range of values of a variable appearing in a basic solution stays within its upper and lower bounds. (See footnotes 1, 2, 3, below and on following page). In this paper it will be assumed that

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1. A problem of Optimum Scheduling of Projects on Punch Card Equipment considered by Clifford Shaw of RAND was characterized by many variables with fixed upper bounds. (These represented the maximum number of hours that could be assigned to a project in a work period.) The method described here was developed to provide a

the reader is familiar with this method. By way of review, a basic solution to (I-1) is defined as a solution in which n-m variables x_j are set equal to zero and the remainder $x_o, x_{j_1}, \ldots, x_{j_m}$ satisfy (I-1) such that $x_{j_i} \geq 0$ and the submatrix $B = [P_o, P_{j_1}, \ldots, P_{j_m}]$ formed from the columns P_{j_i} of coefficients of these variables is non-singular. In each iteration of the simplex algorithm a new basic solution is formed in which one of the basic variables x_{j_r} is replaced by a non-basic variable x_s and one of columns P_{j_r} of the "basis" B is replaced by P_s.

We shall assume that one starts out with a basic solution to the original system which does not violate conditions (I-2). It is possible of course that such a solution may not exist or may be difficult to determine. However, if we begin with phase I of the simplex process where the problem is to determine a basic feasible solution, we will be in the position of having a basic feasible solution to a related problem which satisfies the relations $x_j \leq \alpha_j$.

The regular simplex rules for shifting from one basic solution of the original system (I-1) to the next apply unless the value of a variable in the basic set changes from $x_j \leq \alpha_j$ to a new value

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short cut computing routine and was first reported by Clifford Shaw and the author before the joint RAND-U.C.L.A. Seminar on Industrial Scheduling in the winter of 1952.

2. For other applications of the method described above see A. Charnes and C. Lemke [4].

3. Quite often in linear programming problems it is possible to obtain a starting basis for an enlarged system from a basis of a smaller system. A typical case occurs when a feasible basis for the enlarged system exists which differs from a feasible basis for the original system by having additional rows and corresponding unit vector columns. An example of this technique for bounded variables is found in Charnes [2].

$x_j + \theta y_j > \alpha_j$ or $x_s = \theta > \alpha_s$. Then it is necessary to reduce the permissible range of θ so that $x_j + \theta y_j \leq \alpha_j$ and $x_s \leq \alpha_s$ in order to preserve feasibility in system (I-2). Let $\theta = \theta_o$ be the largest value for which all relations $0 \leq x_j \leq \alpha_j$ are satisfied and suppose for this critical value, θ_o, one of the variables x_t in the basic set $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$ (or the variable x_s being introduced into the basic set) attains its upper bound α_t.

Consider now a new (equivalent) linear programming problem, obtained from the original problem by replacing x_t by $\alpha_t - x_t'$. Letting P_j denote the column of coefficients associated with x_j, the effect is to replace

(a) The constant vector Q by $Q - \alpha_t P_t$

(b) The vector P_t by $-P_t$

(c) The variable x_t by x_t'

(d) Column P_t in the basis is replaced by P_s unless $t = s$; in which case basis vectors are unchanged.

(e) The values of the new basic variables denoted by x_j^* are

$$
\text{(I-3)} \qquad
\begin{aligned}
x_{j_i}^* &= x_{j_i} - \theta_o y_{j_i} \,, && (j_i \neq t) \\
x_s^* &= \theta_o \,, && (s \neq t)
\end{aligned}
$$

where $\theta = \theta_o$ is the critical value of θ. The basic solution of the equivalent programming problem has improved value for the minimizing form (excluding the possibility of degeneracy).

We are now in a position to iterate using the regular simplex algorithm or the modified one as appropriate. It may be of passing

interest to note that the row vector β ("pricing vector") used to determine which vector P_j to introduce into a basis B in subsequent iterations is uneffected whether one (or several columns) of the columns P_{j_i} the basis are replaced by $-P_{j_i}$ since β is defined so that $\beta B = \beta[P_o, P_{j_1}, P_{j_2}, \ldots, P_{j_m}] = [1, 0, \ldots, 0]$. The vector to be introduced into the basis is determined by Min $\beta P_j < 0$ or if x_j has been replaced by x'_j and P'_j is not in the basis, by Min $\beta(-P_j) < 0$ which says a variable x_j at its upper bound may be decreased from its upper bound with improvement in the value of the solution providing the negative of the usual simplex criterion is satisfied.

There is a close relation between the simplex algorithm for the original problem (as modified above for variables with upper bounds) and the one which would be obtained if one were to directly apply the regular simplex procedure to the enlarged problem. The procedure just described was first obtained by noting that when the simplex procedure is applied to the enlarged problem certain computational simplifications could be obtained because the vector associated with x'_j is a unit vector.

II - Block Triangularity (General Case)

By "block" triangular* is meant that if one partitions the matrix of coefficients of the technology matrix into submatrices, the submatrices (or blocks) considered as elements form a triangular

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*A term suggested by Walter Jacobs.

Now the main obstacle toward the full application of standard linear programming techniques to dynamic systems is the magnitude of the matrix for even the simplest situations. For example, a trivial 15-activity--7-item static model, when set up as a 12-period dynamic model, would become a 180-activity by 84-item system, which is considered a large problem for application of the standard simplex method. A fancy model involving, say, 200 activities and 100 items for a static case would become a 2000 x 1000 matrix if recast as a 10-period model. It is clear that dynamic models must be treated with special tools if any progress is to be made toward solutions of these systems.

From a computational point of view, there are a number of observed characteristics of the dynamic models which are often true for static models as well. These are:

(1) The matrix (or its transpose) can be arranged in triangular form (II-2).

(2) Most submatrices A_{ij} are either zero matrices or composed of elements, most of which are zero.

(3) A basis for the simplex method is often block triangular with its diagonal submatrices square and non-singular (referred to as a "square block triangular" basis).

(4) For dynamic models similar type activities are likely to persist in the basis for several periods.

To illustrate, consider a dynamic version of the Leontief model in which (a) alternative activities are permitted; (a simple case

system, (2). For example, von Neumann, [9], in considering a constantly expanding economy, developes a linear dynamic model whose matrix of coefficients may be written in the form (II-1)

$$(\text{II-1}) \quad \begin{bmatrix} A & & & \\ -B & A & & \\ & -B & A & \\ & & -B & \ddots \end{bmatrix} \quad ; \quad (\text{II-2}) \quad \begin{bmatrix} A_{11} & & \\ A_{21} & A_{22} & \\ \cdot & \cdot & \cdot \quad \cdot \\ A_{T1} & A_{T2} & A_{TT} \end{bmatrix}$$

where A is the submatrix of coefficients of activities initiated in period t, and B is the submatrix of output coefficients of these activities in the following period. When activities extend over many time periods, it is not difficult to show that they can be subdivided into a set of interlocked subactivities over time, whose input occurs at time t and output at time t+1. Moreover, it is not necessary that the input output blocks A,B be identical from one time period to the next. Because of the general applicability of such a model, the author in an earlier paper suggests that it is worthwhile to give special attention to the solution of linear programming problems where the matrices are of this form, [1a]. In this paper, however, we shall consider the partitioned form (II-2) as standard; primarily because, cast in this form, most models will involve fewer equations and variables and because the essential feature of block triangularity is preserved.

would be where steel can be obtained from direct production or storage), (b) inputs to an activity for production in the t^{th} time period may occur in same or earlier time periods. It can be shown in this model that (a) a basic solution will have exactly m activities in each time period (where m = number of time dependent equations), (b) each shift in basis will bring in a substitute activity in the same time period, and (c) optimization can be carried out as a sequence of one-period optimization problems; i.e., the optimum choice of activities (but not their amounts) can be determined for the first time period (independent of the later periods); this permits a determination for the second time period (independent of the later periods), etc., [8c].

When flow models are replaced with more complex models which include initial inventories, capacities, and the building of new capacities, the ideal structure of a basis (see third characteristic above) no longer holds. However, tests (carried on since 1950) on a number of cases indicate that bases, while often <u>not square block triangular in the sense above, could be made so by changing relatively few columns in the basis</u> (e.g., one or two activities in small models). This characteristic of <u>near square block</u> triangularity of the basis, i.e., with non-singular square submatrices down the diagonal is, of course, computationally convenient and this paper will be concerned with ways to exploit it.

However, there appears to be a strong alternative possibility which may be built around bases whose non-zero elements either cluster above (or below) the main diagonal. An example of

this approach is found in the Production Smoothing Problem proposed by Jacobs [6]. Selmer Johnson and the author have been able to show that the basis in that problem either does not extend beyond the main diagonal or if it does, it does so by not more than one column. This will be the subject of another paper. <u>These considerations point up the need for further research on the structure of a basis whose columns are drawn from a block triangular matrix.</u>*

The remainder of this section will be concerned with a technique that should reduce materially the size of the computation job when the given model satisfies the first three empirical properties. Extensions can be introduced to take advantage of property (4) also but these are beyond the scope of the present paper.

Let system (II-2) be the form of the matrix of coefficients. Let A_{tt} be a matrix with n_t columns and m_t rows $(t=0,1,\ldots,T)$. Column vectors P_j which have elements in common with A_{tt}, as well as their corresponding variables, will be considered as "belonging to" the t^{th} period. Similarly, the m_t equations which have coefficients in common with A_{tt} are also considered as belonging to the t^{th} period.

We wish to apply the simplex algorithm and will do so for each iteration through the use of an <u>artificial basis</u> $\bar{B}$ and a <u>true basis</u>

*Recently H. Markowitz of RAND has pointed out that inverses for bases composed largely of zero elements (randomly distributed or in blocks) may be avoided. Instead, he conjectures that the direct solution of the associated system of equations for each iteration (as is done in the transportation model, [1c]) can be done at a fraction of the usual computational effort for a general system of this size. He is currently developing an electronic computer code to test this method out on large systems.

B. The artificial basis will be square block triangular. As many columns as possible of $\bar{B}$ will be made the same as B.*

It is assumed that expressions are known which express any column P_α of $\bar{B}$ as a linear combination of the columns P_{j_i} of B.

$$\text{(II-3)} \qquad P_\alpha = \sum_{i=0}^{m} \eta_{i\alpha} P_{j_i} \qquad (P_{j_0} \equiv P_0) ,$$

so that we may write

$$\text{(II-4)} \qquad \bar{B} = B \cdot \eta$$

where η is a matrix whose α^{th} column has coefficients $\eta_{i\alpha}$. If $\bar{B}$ has many columns in common with B, it is clear that η will be composed largely of unit vectors so that the essential part of η consists of those columns which give the representations in terms of P_{j_i} of those vectors P_{α_i} in $\bar{B}$ and not in B (called artificial

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*The use of true and artificial bases has many important applications. Some of these will be considered in subsequent papers. In general, the purpose of the artificial basis is to have a basis in a desirable form for computations; the purpose of the side condition is to transform the computations based on an artificial basis to their correct value in terms of the true basis. Application of such a device calls for some judgment, since it is clear that if the number of side conditions becomes too large, the method will have no advantages over a standard solution.

For our purposes it is not strictly necessary that these artificial vectors be part of the original set of vectors P_j; they could (if convenient) be any vectors (such as the artificial unit vectors found in the regular simplex process).

vectors) in terms of P_{j_1}.

In the standard application of the revised simplex algorithm [8a], it is customary to maintain the inverse of the basis (or the inverse in product form [5]) in order to solve readily the system of equations

$$\text{(II-5)} \qquad \beta B = [1, 0, \ldots, 0]$$

$$\text{(II-6)} \qquad BY = P_s$$

where β is the <u>price vector</u>* and Y is the representation of the vector entering the next basis in terms of the vectors of the preceding basis. This we replace by

$$\text{(II-7)} \qquad \beta\bar{B} = [1, 0, \ldots, 0]\eta$$

$$\text{(II-8)} \qquad \bar{B}\bar{Y} = P_s \quad ; \qquad Y = \eta\cdot\bar{Y} \quad .$$

In order to solve readily systems involving $\bar{B}$ as matrix, it is only necessary to maintain the inverses of the smaller non-singular

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*This definition of β applies only to phase II of the simplex procedure; for phase I, replace the right-hand side by [0,1,0,...,0], see [8a], where it is assumed that the first column of the basis is P_o corresponding to the variable x_o in [I-1]. In phase I, however, the second column corresponds to dummy variable x_{n+1} whose value is being maximized.

diagonal matrices of $\bar{B}$ (or their inverses in product form) which we denote by $\bar{B}_{oo}$, $\bar{B}_{11}$, ... For example, in solving (II-8) the inverse of $\bar{B}_{oo}$, when multiplied by the first m_o components of the right-hand row vector will give the first m_o components of $\bar{Y}$. Because of triangularity, these values can be substituted into equations associated with the second time period and the next m_1 components of $\bar{Y}$ obtained through application of the inverse of $\bar{B}_{11}$, etc.

Except for elements associated with the inverses of the diagonal blocks (either in direct or product form), it will be noted that all other operations involve scalar products with parts of columns of $\bar{B}$ in their original form. These often are entirely null or composed largely of zero elements. This advantage is to be contrasted with the case where the entire artificial basis $\bar{B}^{-1}$ (or B^{-1}) is maintained!

Each change in true basis replaces a vector P_{j_r} by P_s. The representation of P_{j_r} in terms of the vectors of the next basis is given by

$$\text{(II-9)} \qquad P_{j_r} = \sum_{i \neq r} \eta_{ij_r} P_{j_i} + \eta_{rj_r} P_s$$

where η_{ij_r} is given in terms of the components of y_i of Y by

$$\text{(II-10)} \qquad \eta_{ij_r} = -y_i/y_r \ , \quad \eta_{rj_r} = 1/y_r \qquad (i \neq r)$$

By substituting the expression for P_{j_r} in (II-3), vectors P_{α_i} can be reexpressed in terms of the next basis. This constitutes the transformation of η from one cycle to next. If P_{j_r} is in $\overline{B}$ **also, then** in the next cycle P_{j_r} will be an artificial vector, and expression (II-9) forms part of the essential part of η for the next cycle. If P_{j_r} is not in $\overline{B}$, its expression (II-9) in terms of the true basis is no longer of any interest and may be thrown away. In the former case the size of η (where by "size" is meant the number of essential columns) has increased. In the latter it has remained the same. On the other hand if P_s was artificial in $\overline{B}$ in the previous cycle and introduced into B in the next, then the size of η would decrease.

We next consider the possibility of modifying the definition of $\overline{B}$ in order to decrease the number of essential columns in η. This is important because unless the size of η can be kept relatively low, the transformations of η are the same as the transformations on the columns of B^{-1} in the standard simplex algorithm and no advantage would accrue by using this approach. Let P_β be a column of B, not in $\overline{B}$, belonging to period t, and suppose there are artificial vectors also belonging to this period. We form the representation of the partitioned part of P_β belonging to period t in terms of the columns of $\overline{B}_{tt}$

(II-11) $\quad P_\beta^{(t)} = \overline{B}_{tt}\ \overline{Y}^{(t)}$

where those components of a vector associated with period t are denoted by a superscript (t). Consider now those components of $\overline{Y}^{(t)}$, corresponding to artificial column vectors P_α. If any of these components are non-zero, then it is clear that P_β can replace the corresponding P_α in $\overline{B}$ and thus reduce the size of the essential part of η; moreover, $\overline{Y}^{(t)}$ may be used as in regular simplex routine to obtain the corrected inverse of $\overline{B}_{tt}^{-1}$ (in direct or product form).

It will be noted that, because of block triangularity of the original matrix, the first step in the representation of P_s in terms of columns of B by (II-8) gives the representation of $P_s^{(t)}$ in terms of the columns of $\overline{B}_{tt}$ (where P_s belongs to period t). However, this is the relation required by (II-11) and may be used to determine if P_s can replace an artificial vector in $\overline{B}$. Hence, in this case it is not necessary to make a separate calculation to determine whether P_β can replace P_α in $\overline{B}$. Moreover, by saving the relations that express $P_s^{(t)}$ in terms of the columns of $\overline{B}_{tt}$ (and transforming them, should the columns of $\overline{B}_{tt}$ change) it is possible to have all relations (II-11) readily available. There are, of course, as many such relations as there are essential columns in η; however, the work involved in maintaining these relations is a fraction of that required for η.

III - Secondary Constraints

Here, as in the special case of variables with upper bounds, we suppose in addition to (I-1) the variables must satisfy a

"secondary" system of constraints of the form

$$\text{(III-1)} \quad \sum_{j=1}^{n} a_{kj}x_j + x_{n+k} = b_k \qquad (k = 1,2,\ldots,\ldots,m'),$$
$$x_{n+k} \geq 0 \;;$$

mathematically, these latter restrictions differ in no way from the former, but from the physical point of view it is anticipated that only a small subset of the restrictions (III-1) will be <u>active</u> in an optimal solution. By a constraint being "active" is meant that its "slack" variables $x_{n+k} = 0$.

To illustrate: In a gasoline blending problem there may be a number of equations controlling the performance characteristics of a blend, for example, its viscosity, specific gravity, ... It is expected that only one or two of these characteristics are limiting in a given run. A second related example would be <u>capacity restraints</u> on various processes within the refinery as in a thermal cracking unit, storage capacity of tanks, etc. Again it may be expected that <u>some</u> of these restraints will be active, but <u>not many</u> for any particular problem on any particular iteration of the simplex process.

To take advantage of the expected small number of active secondary constraints, we shall make use (as in section II) of a <u>true basis</u> and of an <u>artificial square block triangular basis</u> in executing the simplex algorithm.

For this problem the artificial basis will have a special form; its columns will consist of vectors P_o, P_{β_1}, P_{β_2}, ..., P_{β_m};

P_{n+1}, P_{n+2}, ..., $P_{n+m'}$ where $1 \leq \beta_i \leq n$. The vectors P_{n+i} are obviously unit vectors. $\bar{B}$ may be partitioned in the form

$$\text{(III-2)} \qquad \bar{B} = \begin{bmatrix} \bar{B}^{(1)} & . \\ \bar{B}^{(2)} & I_{m'} \end{bmatrix} \qquad \cdots$$

where $\bar{B}_m^{(1)}$ are the components associated with system (I-1), $\bar{B}^{(2)}$ with system (III-1), and $I_{m'}$ is an m' x m' identity matrix. Since (III-2) is in "square block" triangular form, it is clear that it is only necessary to know the inverse of the smaller m x m submatrix $\bar{B}^{(1)}$ (or its equivalent) to determine Y or β through η.

Let the true basis consist of the vectors P_o, P_{j_1}, ..., $P_{j_{m+m'}}$. It will be supposed that many of the vectors of the artificial basis are the same as those in the true. However, some of the unit vectors in the artificial basis may not be found in the true basis. In this case, the side conditions (II-3) are simply the representations of these artificial unit vectors in terms of the vectors of the true basis. With these observations one may now proceed to apply the algorithm of section II to the special case of secondary constraints.

The method of additional restraints has been found to be a powerful tool in solving many large scale systems, for example, the recent successful solution of a 49-city traveling salesman problem. Indeed, in some problems it is easier to find a point

p where a function assumes a minimum for a region R* than for a smaller required region R in R*. If, by good luck, the minimum value of the function is assumed at a point p of R* which is also a point of R, then it is obvious that this point is also the required optimum solution for R.

To illustrate, suppose an optimum solution is obtained for (I-1) without regard to whether the slack variables x_{n+i} defined by the secondary constraint conditions (III-1) are non-negative or not. If by luck all $x_{n+i} \geq 0$ then the solution is an optimal solution for (I-1) and (III-1) combined. Consider now a situation where only a few of the secondary restraints are expected to be active; i.e., the solution may be expected to have some but only a few negative values. In this case it appears reasonable to take the optimum solution of (I-1) as a good starting point to begin a corrective procedure to clean up the negative values of the variables in order to obtain an optimum solution for (I-1) and (III-1) combined. This approach can be formalized as follows:

(a) Find an optimum basic solution $(x_o, x_{j_1}, x_{j_2}, \ldots, x_{j_m})$ to the smaller system (I-1) without regard to (III-1). (The fact that a smaller system is being first solved can be an important computational advantage in practice over methods which work always with the entire system.)

(b) Determine the values of the slack variables by substitution in (III-1) and regard $B = [P_o, P_{j_1}, \ldots, P_{j_m}; P_{n+1}, \ldots, P_{n+m}]$ as a starting basis for the combined system (I-1) and (III-1).

If $x_{n+1} \geq 0$ for the solution associated with this basis, then this is an optimal solution for combined system and this terminates the algorithm. If not, it is clear that the "price vector" β for this basis satisfies $\beta P_j \geq 0$ for $j=1,2,\ldots,n,n+1,\ldots,n+m'$. Indeed, β has clearly zero components for those components corresponding to equations of system (III-1) while those corresponding to system (I-1) are the same as the optimal β for the smaller system.

(c) This basis can be used to initiate the <u>dual simplex algorithm</u> of Lemke, as this method requires that all $\beta P_j \geq 0$. See [7], [8b]. Computationally, the latter procedure resembles the simplex algorithm in that it shifts from one basis to the next and differs only in the selection criteria for determining the column to drop and to introduce into the next basis. It does not require that x_{j_i} remain non-negative from one iteration to the next.

(d) The initial basis B is block triangular; however, subsequent basis will have to lose this characteristic. However by introducing an artificial basis as given in (III-2) and side conditions to express the artificial unit vectors in terms of the vectors of the true basis, the computational advantages discussed in section II of the near block triangularity of B can also be realized.

Finally, it should be remarked that if it is known in advance that a certain variable x in a general linear programming must be a basic variable in the optimum solution, this variable can be eliminated from all but one of the equations and the objective function. The resulting smaller system can then be optimized and the values of this solution substituted in the remaining equation (which is treated like a secondary constraint equation) to determine the value x. To illustrate in a dynamic economic model it is highly likely that the stock production activities will be operating in all time periods. In a recent example H. M. Wagner* has shown that it is convenient to eliminate the corresponding variables from all but one equation each and optimize the remaining system. In a few examples tested by this approach it turned out to be a trivial matter to make proper adjustments to correct the negative stock production activities even without recourse to the dual simplex algorithm recommended earlier.

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